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Reduction of Poisson algebras at nonzero momentum values

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Abstract

A reduction of a Poisson manifold using the ideal $I(J)$ generated by the momentum map was introduced by Śniatycki and Weinstein (1983). This reduction has been extended to nonzero momentum values μ by two methods: by shifting to zero momentum on a larger space, the product with the coadjoint orbit; and by the method of Wilbour and Kimura (1991, 1993) using the modified ideal $I(J - \mu)$. It is shown that these two methods produce isomorphic reduced algebras under the assumptions that the symmetry group is connected and that the stabilizer group of μ also is connected. If the latter assumption fails, the shifted reduced algebra is isomorphic to a (possibly proper) subalgebra of the Wilbour–Kimura algebra.

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1. Introduction

Classically, the goal of reduction of a symplectic manifold with symmetry is to produce a smaller symplectic manifold. Specifically, suppose there exists an equivariant momentum map for the group action (i.e. the symmetry). The level set of a regular value of the momentum map is a submanifold, and if the group action on this level set is fibrating, then the orbit space of the level set inherits a natural symplectic structure. This is the (geometric) reduction of Meyer [10], and Marsden and Weinstein [9].

This basic reduction has been generalized and extended in a number of ways. (For a survey see [13] and references therein.) One of the earliest was a construction of Śniatycki and Weinstein [11]. Their approach is reminiscent of algebraic geometry in that it works with the Poisson algebra of functions rather than with the manifold, and produces a reduced algebra but no reduced space. Śniatycki and Weinstein showed that at least in some cases,

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this algebra carries information about the geometric quantization of the system that is lost by more geometric methods of reduction. More recent work has shown that this reduced algebra is isomorphic to the zero-dimensional classical BRST cohomology [2].

For this algebraic reduction, the original definition applied only to the zero value of momentum – or, by a trivial extension, to values fixed by the coadjoint action. Geometrically, the difficulty is that some nonzero level sets are not coisotropic. (The corresponding algebraic difficulty is described in Section 2.) Two methods for handling the difficulty have been developed. The "shifting trick" transforms the problem of reduction at a nonzero value of momentum to reduction at zero in a larger space. When this method is applied to the geometric reduction, it produces results isomorphic to those of the usual Meyer, Marsden–Weinstein (MMW) reduction [3, Section 26; 7].

A second method, appearing in work of Wilbour [12,13] and Kimura [5], works within the function algebra on the original space. The main result of the present paper is that under relatively mild assumptions, the shifting trick and the Wilbour–Kimura method produce isomorphic reduced algebras. The assumptions are that the original group is connected and that the stabilizer subgroup of the momentum value (under the coadjoint action) also is connected. When the latter assumption fails, a modified version of the shifting trick yields a reduced algebra isomorphic to the Wilbour–Kimura algebra.

Section 2 establishes notation and describes the algebraic reduction, the Wilbour–Kimura construction, and the shifting trick. Section 3 states the main result precisely, and discusses the main ideas used in the proofs. The proofs themselves appear in Section 4.

2. Notation and basic constructions

This section describes the algebraic reduction for zero momentum values. Then two methods of extending this reduction to nonzero values are presented: the shifting method and the Wilbour–Kimura (WK) method.

Let $(\mathcal{P}, \{\ ,\ \})$ be a Poisson manifold, and suppose that there is a Hamiltonian action of a connected Lie group G on $\mathcal{P}$; that is, an action by Poisson maps with equivariant map J. The action will be variously denoted by $\Phi(g, p) = \Phi_g(p) = g \cdot p$. For ξ in the Lie algebra $\mathfrak{g}$ of G, let $J_\xi = \langle J, \xi \rangle$, where the brackets $\langle\ ,\ \rangle$ indicate the evaluation of $\mathfrak{g}^*$ on $\mathfrak{g}$. Recall that the defining property of a momentum map is that for all $A \in C^\infty(\mathcal{P})$,

$$\{A, J_\xi\} = \mathrm{d}A(\xi_p),$$

where ξ_p is the infinitesimal generator on $\mathcal{P}$ corresponding to ξ. Frequently $\xi_i, i = i, \ldots, k$, will be a basis for $\mathfrak{g}$, and then $J_i = \langle J, \xi_i \rangle$. The (commutative) ideal generated by the components of the momentum map is

$$I(J) := \{A \in C^\infty(\mathcal{P}) \mid A = A^i J_i \text{ for some } A_i \in C^\infty(\mathcal{P})\}.$$

Note that $I(J)$ is independent of the choice of basis for $\mathfrak{g}$. (The Einstein summation convention on repeated indices will hold except where explicit summation notation indicates a sum over a subset of the indicial values.)

For any (commutative) ideal I, define the Poisson normalizer of I to be

$$N(I) := \{A \in C^\infty(\mathcal{P}) \mid \{A, I\} \subseteq I\}.$$

The Poisson structure on $C^\infty(\mathcal{P})$ induces a Poisson structure on

$$\frac{N(I)}{[I \cap N(I)]}. \tag{1}$$

Equivariance of J implies that the ideal $I(J)$ is contained within its normalizer, so there is a natural Poisson structure on the quotient

$$\frac{N(I(J))}{I(J)}. \tag{2}$$

This quotient algebra is the result of the algebraic reduction of [11]. (Actually, the definition in the reference is slightly different: it considers G-invariant equivalence classes in $C^\infty(\mathcal{P})/I(J)$. But when G is connected, as is assumed in [11], this is equivalent to the definition (2).) If zero is a regular (or weakly regular) value of J, then the algebra (2) is naturally isomorphic to the algebra of smooth functions on the reduced space $J^{-1}(0)/G$ [1,11]. Restricting to the constraint set $J^{-1}(0)$ in constructing the reduced space corresponds to taking the quotient by $I(J)$ in (2).

Now consider a constraint set $J^{-1}(\mu)$ for a fixed nonzero value $\mu \in \mathfrak{g}^*$. Let $K = J - \mu$, so $K_\xi = \langle K, \xi \rangle$ and $K_i = \langle K, \xi_i \rangle$. Replace $I(J)$ by

$$I(K) := \{A \in C^\infty(\mathcal{P}) \mid A = A^i K_i \text{ for some } A^i \in C^\infty(\mathcal{P})\}.$$

The ideal $I(K)$ may not be contained in its normalizer $N(I(K))$, even for regular values of μ. (If the coadjoint orbit of μ is nontrivial then $J^{-1}(\mu)$ fails to be coisotropic. The latter implies that $N(I(K))$ does not contain $I(K)$.) In this case we use (1) to define the reduced algebra

$$\mathcal{A}_K := \frac{N(I(K))}{[I(K) \cap N(I(K))]}. \tag{3}$$

If μ is a regular value, then $I(K))$ is the ideal of functions which vanish on $J^{-1}(\mu)$. In this case Wilbour [12] (see also [5]) has shown that $\mathcal{A}_K$ is the algebra of functions on the MMW reduced space $J^{-1}(0)/G_\mu$, where G_μ is the stabilizer subgroup of μ under the coadjoint action.

Earlier work handled the complications of nonzero values of momentum by a technique called "shifting". For a fixed value μ in $\mathfrak{g}^*$, shifting uses the coadjoint orbit $\mathcal{O}$ of μ in $\mathfrak{g}^*$. As above, a centered dot will denote an action on the left; thus $g \cdot \nu$ denotes the coadjoint action by g^{-1} on ν. There is a natural Poisson structure on $\mathcal{O}$ given by

$$\{\xi, \zeta\}^{\mathcal{O}}(\nu) = \langle \nu, [\xi, \zeta] \rangle,$$

where ξ and ζ are in $\mathfrak{g}$ and thus are linear functions on $\mathfrak{g}^*$. (This is the Poisson structure induced by the canonical (up to sign) symplectic form on $\mathcal{O}$ [8].)

Define a new Poisson manifold $(\widetilde{\mathcal{P}}, \{\ ,\ \}^{\sim})$ as follows. Let $\widetilde{\mathcal{P}} = \mathcal{P} \times \mathcal{O}$, with a typical point given by (p, ν) and projections $\widetilde{\pi}_1(p, \nu) = p$ and $\widetilde{\pi}_2(p, \nu) = \nu$. Also define injections $i_\nu : \mathcal{P} \to \widetilde{\mathcal{P}}$ and $i_p : \mathcal{O} \to \widetilde{\mathcal{P}}$ by $i_\nu(p) = (p, \nu) = i_p(\nu)$. Then the Poisson bracket is given by

$$\{\widetilde{A}, \widetilde{B}\}^{\sim}(p, \nu) = \{\widetilde{A} \circ i_\nu, \widetilde{B} \circ i_\nu\}(p) - \{\widetilde{A} \circ i_p, \widetilde{B} \circ i_p\}^{\mathcal{O}}(\nu).$$

(If $\mathcal{P}$ is a symplectic manifold, then the symplectic structure on $\widetilde{\mathcal{P}}$ corresponding to the Poisson bracket $\{\ ,\ \}^{\sim}$ is the difference of the pullbacks of the symplectic forms on $\mathcal{P}$ and $\mathcal{O}$.)

The actions on $\mathcal{P}$ and $\mathcal{O}$ induce an action $\widetilde{\Phi}$ on $\widetilde{\mathcal{P}}$; the (equivariant) momentum map for this action is $\widetilde{J} = J \circ \widetilde{\pi}_1 - \widetilde{\pi}_2$, i.e. $\widetilde{J}(p, \nu) = J(p) - \nu$. The basic idea of the "shifting trick" is to use the zero value for $\widetilde{J}$ in place of the value μ for J. For instance, in the MMW reduction, $\widetilde{J}^{-1}(0)/G$ can be identified in a natural way with $J^{-1}(\mu)/G_\mu$. For the algebraic reduction, define the "shifted reduced algebra" by

$$\mathcal{A}_S := \frac{N(I(\widetilde{J}))}{I(\widetilde{J})}. \tag{4}$$

3. Statement and discussion of results

The main result of this paper is the following.

Theorem 1. *Suppose that G is a connected Lie group and that there exists a Hamiltonian action of G on the Poisson manifold $(\mathcal{P}, \{\ ,\ \})$. If G_μ is connected, then at momentum value μ there is a Poisson algebra isomorphism between the Wilbour–Kimura reduced algebra $\mathcal{A}_K$ in* (3) *and the shifted reduced algebra $\mathcal{A}_S$ in* (4).

To identify key ideas in the construction of the isomorphism, consider first the special case in which G_μ is compact. Let A be a function in $N(I(K))$. Proposition 3 proves that each equivalence class in $\mathcal{A}_K$ includes a G_μ-invariant function, so assume A is G_μ-invariant. Let B be the principal G_μ-bundle $(G, \mathcal{O}, G_\mu)$, and let s be a local section of B; then

$$s(\nu) \cdot \mu = \nu,$$

and we can identify ν with the coset $s(\nu)G_\mu$. Define

$$\widetilde{A}(p, \nu) = A(s(\nu)^{-1} \cdot p). \tag{5}$$

By the G_μ-invariance of A, the extended function $\widetilde{A}$ is well-defined independent of the choice of section s (because at each ν the possible choices of section differ by an element of G_μ). Furthermore $\widetilde{A}$ is G-invariant on $\widetilde{P}$, because $s(g \cdot \nu) \cdot \mu = g \cdot \nu = g \cdot s(\nu) \cdot \mu$, so $s(g \cdot \nu)^{-1} \cdot g \cdot s(\nu) \in G_\mu$ and

$$\begin{aligned}\widetilde{A}(g \cdot (p, \nu)) &= \widetilde{A}(g \cdot p, g \cdot \nu) \\ &= A(s(g \cdot \nu)^{-1} \cdot g \cdot p) = A(s(\nu)^{-1} \cdot p) = \widetilde{A}(p, \nu).\end{aligned}$$

Therefore $\widetilde{A}$ is in $N(I(\widetilde{J}))$, and it can be shown that the map from A to $\widetilde{A}$ induces the required isomorphism.

If G_μ is not compact, then there may be no G_μ-invariant representative of the equivalence class. However, if B is a trivial bundle, then we may fix a global section s and define the extension $\widetilde{A}$ by (5). Roughly speaking, functions in $N(I(K))$ are those which are (locally) constant on the intersection of orbits with $J^{-1}(\mu) = K^{-1}(0)$: by definition $A \in N(I(K))$ if and only if

$$\{A, J\} = \{A, K\} \in I(K), \tag{6}$$

so in particular

$$\{A, J\} = 0 \text{ on } K^{-1}(0). \tag{7}$$

(In some cases $I(K)$ is a proper subset of the ideal of functions that vanish on $K^{-1}(0)$. Then (6) is stronger than (7), but the approximate statement suffices for the present heuristic discussion.) If $(p, \nu) \in \widetilde{J}^{-1}(0)$, by equivariance $J(s(\nu)^{-1} \cdot p) = s(\nu)^{-1} \cdot \nu = \mu$, i.e. $s(\nu)^{-1} \cdot p \in K^{-1}(0)$. The G_μ-invariance of A on $K^{-1}(0)$ and an argument similar to that of the preceding paragraph, but restricted to $\widetilde{J}^{-1}(0)$, show that the extension $\widetilde{A}$ in (5) is constant on G-orbits within that level set. This suffices in e.g. the nonsingular case to show that $\widetilde{A} \in N(I(\widetilde{J}))$, as before inducing the desired isomorphism.

The proof of the main result combines the ideas of these two special cases. The structure group of the principal bundle B can be reduced to a maximal compact subgroup H of G_μ. If G_μ is connected, then H is, also. The reduction of the bundle implies that a set $\mathcal{S}$ of local sections of the original bundle can be chosen so that their domains cover the base space $\mathcal{O}$ and the associated transition functions take their values in H. The proofs below show the following. If H is connected then each equivalence class in $\mathcal{A}_K$ contains an H-invariant representative. For such invariant functions, the extension defined in (5) gives an element of $N(I(\widetilde{J}))$. The extension induces a Poisson map from $\mathcal{A}_K$ to $\mathcal{A}_S$, which is shown to be an isomorphism. Furthermore, although the extension procedure (5) depends on the choice of sections, the induced map between the reduced algebras is independent of this choice.

If G_μ is not connected, then the maximal compact subgroup H also may be disconnected, and there may be classes in $\mathcal{A}_K$ that do not include an H-invariant representative. In this case it is possible to identify $\mathcal{A}_K$ with the results of a modified shifting construction. Let $G_{\mu 0}$ be the identity component of G_μ and replace G_μ by $G_{\mu 0}$, the orbit $\mathcal{O}$ by $G/G_{\mu 0}$, and H by a maximal compact subgroup of $G_{\mu 0}$ in all constructions and proofs. Then $\mathcal{A}_K$ is identified with a quotient of function spaces on $\mathcal{P} \times (G/G_{\mu 0})$. The latter is a covering space of $\widetilde{\mathcal{P}}$. The momentum on $\mathcal{P} \times (G/G_{\mu 0})$ is the composite of $\widetilde{J}$ with the projection π of the covering. The modified proofs give an isomorphism of $\mathcal{A}_K$ with $N(I(\widetilde{J} \circ \pi))/I(\widetilde{J} \circ \pi)$. There is a natural inclusion of $I(\widetilde{J})$ into $I(\widetilde{J} \circ \pi)$ and of $N(I(\widetilde{J}))$ into $N(I(\widetilde{J} \circ \pi))$. Furthermore if $\widetilde{A} \circ \pi \in I(\widetilde{J} \circ \pi)$, then $\widetilde{A}$ is in $I(\widetilde{J})$ "locally" because π is a Poisson covering map. The partition of unity argument before Lemma 5 then shows that $\widetilde{A} \in I(\widetilde{J})$ globally. This means that the intersection of $I(\widetilde{J} \circ \pi)$ with the image of $N(I(\widetilde{J}))$ equals the image of $I(\widetilde{J})$, so $\mathcal{A}_S$ can be injected into $\mathcal{A}_K$. An example can be constructed with $G = Sl(2, \mathbb{R})$ for which the inclusion is proper.

4. Proofs

Composing a function in $N(I(K))$ with a translation by $g \in G$ yields another function in $N(I(K))$. In later calculations, it will be important to know the exact form of the dependence on g of the Poisson bracket of such functions. This dependence is articulated in the following lemma.

Lemma 2. *Let ξ be a fixed element of $\mathfrak{g}$. If $A \in N(I(K))$, then there are smooth functions A^i, $i = 1, \ldots, k$, on $\mathcal{P} \times G$ such that*

$$\{A \circ \Phi_g, K_\xi\}(p) = \{A \circ \Phi_g, J_\xi\}(p) = A^i(p, g) K_i \circ \Phi_g(p).$$

(Note that the Poisson bracket is computed on $\mathcal{P}$, i.e. with g held constant.) Also there are smooth functions B^i, $i = 1, \ldots, k$, on $\mathcal{P} \times G$ such that if $g \in G_\mu$, then

$$\{A \circ \Phi_g, K_\xi\}(p) = \{A \circ \Phi_g, J_\xi\}(p) = B^i(p, g) K_i(p),$$

so that $A \circ \Phi_g \in N(I(K))$.

Proof. In each equation, the first equality holds because J and K differ by a constant in $\mathfrak{g}^*$. By the properties of group actions and momentum maps,

$$\{A \circ \Phi_g, J_\xi\}(p) = \mathrm{d}A(\mathrm{d}\Phi_g[\xi_P(p)]) = \mathrm{d}A[(Ad_g\xi)_P(g \cdot p)] = \{A, J_{Ad_g\xi}\}(g \cdot p).$$

There are smooth functions C^i on $G \times \mathfrak{g}$ such that

$$Ad_g\xi = C^i(g, \xi)\xi_i.$$

Combining the preceding equations and the fact $A \in N(I(K))$, we have the following:

$$\begin{aligned}\{A \circ \Phi_g, J_\xi\}(p) &= C^i(g, \xi)\{A, J_i\}(g \cdot p)\\ &= C^i(g, \xi)\{A, K_i\}(g \cdot p)\\ &= C^j(g, \xi) D^i_j(g \cdot p) K_i(g \cdot p).\end{aligned}$$

Recall that ξ is fixed. Define $A^i(g, p) = C^j(g, \xi) D^i_j(g \cdot p)$ to obtain the first conclusion.

For $g \in G_\mu$, by equivariance $K_i \circ \Phi_g = g \cdot K_i = \langle K, Ad_{g^{-1}}\xi_i\rangle$. Thus $K_i \circ \Phi_g(p) = C^j(g^{-1}, \xi_i) K_j$. The second conclusion follows with $B^j(g, p) = C^j(g^{-1}, \xi_i) A^i(p, g)$. □

Let H be a connected compact subgroup of G_μ. Pick a bi-invariant Riemannian metric on H with total volume equal to one. For $A \in C^\infty(\mathcal{P})$, define the averaged function $\bar{A}$ by

$$\bar{A}(p) = \int_H A(h \cdot p)\,\mathrm{d}h,$$

where "dh" represents the bi-invariant volume form.

Proposition 3. *If $A \in N(I(K))$ then A and $\bar{A}$ belong to the same equivalence class in $\mathcal{A}_K$.*

Proof. From Lemma 2 it follows immediately that $\bar{A}$ is also in $N(I(K))$:

$$\{\bar{A}, K_\xi\} = \int_H \{A \circ \Phi_h, K_\xi\}\,\mathrm{d}h = \int_H B^i(p,h)K_i(p)\,\mathrm{d}h = \left[\int_H B^i(p,h)\,\mathrm{d}h\right]K_i(p).$$

It remains to show that $A - \bar{A} \in I(K)$. Note that

$$A - \bar{A} = \int_H (A - A \circ \Phi_h)\,\mathrm{d}h.$$

We will see that the integrand is a ($C^\infty(\mathcal{P})$-) linear combination of generators of $I(K)$, with coefficients that are integrable functions of $h \in H$. To do so, it is convenient to show that these functions are continuous (in fact smooth) on a subset N that differs from H by a set of measure zero. We have a bi-invariant Riemannian metric on H. For each vector ξ in the unit sphere $T_e^1 H$ at the identity e, let $m(\xi)$ be the distance from e to the cut point along the geodesic $\exp(s\xi)$. Let $E = \{s\xi : \xi \in T_e^1 H \text{ and } 0 < s < m(\xi)\}$, and let $N = \exp(E)$. By [6, Vol. II, Theorem 7.4, p.100], H is a disjoint union of the open set N, the cut locus, and $\{e\}$. Omitting e and the cut locus has no effect on the integral, as the cut locus has measure zero. (It can be identified with the graph of m, which is a continuous function [6, Vol. II, p.98].) Thus,

$$A - \bar{A} = \int_N (A - A \circ \Phi_h)\,\mathrm{d}h.$$

The value of $A - A \circ \Phi_h$ at $h = \exp(s\xi)$ can be found by integrating along

$$h(t) = \exp(t\xi), \quad 0 \le t \le s,$$

a curve in N from e to $\exp(s\xi)$. The derivative of $A - A \circ \Phi_h$ along this curve is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(A - A \circ \Phi_{h(t)})|_{t=t_0} &= -\frac{\mathrm{d}}{\mathrm{d}t}(A \circ \Phi_{h(t_0)} \circ \Phi_{h(t-t_0)})|_{t=t_0} \\ &= -\{A \circ \Phi_{h(t_0)}, J_\xi\} = -B^i(p, h(t_0))K_i(p)\end{aligned}$$

by Lemma 2. Note that $A - A \circ \Phi_{h(t)} = 0$ at $t = 0$, so integrating along $h(t)$ shows that $(A - A \circ \Phi_h)$ is in $I(K)$. Therefore its integral over N is also in the ideal; i.e. $A - \bar{A} \in I(K)$. □

Proposition 3 is a generalization of Proposition 5.12 in [1], and the proof just given avoids the power series argument used in the reference.

The hypothesis that H is connected may be weakened; the proof still applies if there is a connected subgroup of G_μ containing H. An example shows that the latter condition is necessary. For some values of μ in $sl(2, \mathbb{R})^*$, the isotropy subgroup G_μ has two connected components. One may pick a two-element subgroup of H, one element in each component of G_μ, for which Proposition 3 fails (and the main result must be modified as outlined at the end of Section 3).

In the case of interest, H is a maximal compact subgroup of a connected G_μ. The considerations of the preceding paragraph are then moot, because H also is connected. The next proposition reviews this and other properties of this case.

Proposition 4. *Fix H to be a maximal compact subgroup of G_μ. Then H is connected. Let B be the bundle $(G, \mathcal{O}, G_\mu)$. Then there is a set $\mathcal{S}$ of sections of B satisfying the following conditions:*
(a) $\bigcup_{s\in\mathcal{S}}$ domain$(s) = \mathcal{O}$.
(b) *If $s \in \mathcal{S}$ and $\mu \in$ domain(s) then $s(\mu) = e$, the identity in G.*
(c) *The transition functions for $\mathcal{S}$ take their values in H.*

Proof. By assumption G_μ is connected, so H is, also [4, Theorem 6]. The structure group of B can be reduced to H [6, Vol. I, p.59]; that is, there is an embedded subbundle with structure group H [6, Vol. I, p.53]. By composing with the right action of a single element of G_μ, if necessary, this subbundle may be chosen to include e. Sections of the subbundle automatically satisfy c, and can be chosen to satisfy a and b. □

Let $N(I(K))^H$ indicate the H-invariant functions in $N(I(K))$. Fix a set $\mathcal{S}$ as in Proposition 4, and define

$$\psi : N(I(K))^H \to C^\infty(\widetilde{\mathcal{P}})$$

by

$$\psi(A)(p, \nu) = A(s(\nu)^{-1} \cdot p). \tag{8}$$

By the H-invariance of A and Proposition 4(c), the value of $\psi(A)$ given by (8) is independent of the choice of section $s \in \mathcal{S}$.

If ψ is to give the desired isomorphism of the reduced algebras, then the image of ψ must lie in $N(I(\widetilde{J}))$. This is a global statement on $\widetilde{\mathcal{P}}$, which presents some difficulty because formula (8) gives $\psi(A)$ only locally. Fortunately, it suffices to compute locally, because $I(\widetilde{J})$ is finitely generated (by $\widetilde{J}_1, \ldots, \widetilde{J}_k$). Given a function $\widetilde{A}$ in $C^\infty(\widetilde{\mathcal{P}})$, suppose that every point $\widetilde{p}$ in $\widetilde{\mathcal{P}}$ has a neighborhood $\widetilde{\mathcal{U}}$ on which there are functions $\widetilde{A}^i \in C^\infty(\widetilde{\mathcal{U}})$ such that

$$\widetilde{A}|_{\widetilde{\mathcal{U}}} = \widetilde{A}^i \widetilde{J}_i|_{\widetilde{\mathcal{U}}}. \tag{9}$$

Let the phrase "$\widetilde{A}$ is in $I(\widetilde{J})|_{\widetilde{\mathcal{U}}}$" or "$\widetilde{A}$ is in $I(\widetilde{J})$ locally (on $\widetilde{\mathcal{U}}$)" describe the situation in (9). A partition of unity subordinate to the collection of neighborhoods $\widetilde{\mathcal{U}}$ allows one to patch together the local functions $\widetilde{A}^i$, thus showing that $\widetilde{A} \in I(\widetilde{J})$; that is, if $\widetilde{A}$ is in $I(\widetilde{J})$ locally on some neighborhood of every point, then $\widetilde{A}$ is in $I(\widetilde{J})$ (globally). The necessary local formulas are given by the following lemma; then Proposition 6 gives the global result.

Lemma 5. *Let s be a fixed local section of B with domain $\mathcal{U} \subseteq \mathcal{O}$. Let ξ be a fixed element of $\mathfrak{g}$. Then the following statements hold for functions of $(p, \nu) \in \widetilde{\mathcal{U}} = \mathcal{U} \times \mathcal{P} \subseteq \widetilde{\mathcal{P}}$:*
(a) $K_\xi \circ \Phi_{s(\nu)^{-1}}(p) \in I(\widetilde{J})|_{\widetilde{\mathcal{U}}}$.

(b) *Suppose* $A \in N(I(K))$. *Then*

$$\{A \circ \Phi_{s(\nu)^{-1}}, J_\xi\}(p) \in I(\widetilde{J})|_{\widetilde{\mathcal{U}}}.$$

(c) *Let* $A \in C^\infty(\mathcal{P})$, *and define* $F_p = \psi(A) \circ i_p$, *that is*

$$F_p(\nu) = A(s(\nu)^{-1} \cdot p) \quad (\textit{for } \nu \in \mathcal{U}).$$

Regard ξ *as a function on* $\mathfrak{g}^*$. *Then*

$$\{F_p, \xi\}^{\mathcal{O}}(\nu) \in I(\widetilde{J})|_{\widetilde{\mathcal{U}}}.$$

Proof.

(a) Recall that the fibers of B are cosets of G_μ, and (for any section s of B) $s(\nu) \cdot \mu = \nu$. By the definition of K and the equivariance of J,

$$\begin{aligned}\langle K \circ \Phi_{s(\nu)^{-1}(p)}, \xi\rangle &= \langle J(s(\nu)^{-1} \cdot p) - \mu, \xi\rangle = \langle s(\nu)^{-1} \cdot (J(p) - \nu), \xi\rangle \\ &= \langle J(p - \nu), Ad_{s(\nu)}\xi\rangle = \langle \widetilde{J}(p, \nu), Ad_{s(\nu)}(\xi)\rangle \in I(\widetilde{J})|_{\widetilde{\mathcal{U}}}.\end{aligned}$$

(b) The Poisson bracket on $\mathcal{P}$ is computed with ν held constant. Thus Lemma 2 applies to give

$$\{A \circ \Phi_{s(\nu)^{-1}}, J_\xi\}(p) = A^i(p, s(\nu)^{-1})K_i \circ \Phi_{s(\nu)^{-1}}(p),$$

which is in $I(J)|_{\widetilde{\mathcal{U}}}$ by part (a).

(c) The Hamiltonian vector field of the function ξ on $\mathfrak{g}^*$ generates the flow $\exp(t\xi) \cdot \nu$ on $\mathcal{O}$. Therefore

$$\begin{aligned}\{F_p, \xi\}^{\mathcal{O}}(\nu) &= \frac{\mathrm{d}}{\mathrm{d}t} F_p(\exp(t\xi) \cdot \nu)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} A([s(\exp(t\xi) \cdot \nu)]^{-1} \cdot p)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} A \circ \Phi_{s(\nu)^{-1}}(s(\nu) \cdot [s(\exp(t\xi) \cdot \nu)]^{-1} \cdot p)|_{t=0} \\ &= \{A \circ \Phi_{s(\nu)^{-1}}, J_\zeta\}(p),\end{aligned}$$

where $\zeta = (\mathrm{d}/\mathrm{d}t)s(\nu)[s(\exp(t\xi) \cdot \nu)]^{-1}|_{t=0}$. By part (b), this lies in $I(\widetilde{J})|_{\widetilde{\mathcal{U}}}$. □

Proposition 6. *If* $A \in N(I(K))^H$, *then* $\psi(A) \in N(I(\widetilde{J}))$.

Proof. It suffices to show that $\{\psi(A), \widetilde{J}_\xi\}^\sim$ is in $I(\widetilde{J})$. Note that

$$\widetilde{J}_\xi \circ i_\nu = J_\xi - \langle \nu, \xi\rangle = J_\xi + \text{constant}$$

and

$$\widetilde{J}_\xi \circ i_p = J_\xi(p) - \xi = -\xi + \text{constant}$$

(interpreting ξ as a function on $\mathfrak{g}^*$). Also observe from (8) that

$$\psi(A) \circ i_\nu = A \circ \Phi_{s(\nu)^{-1}},$$

and that $\psi(A) \circ i_p$ is the function F_p in Lemma 5(c). Thus by definition of the Poisson bracket on $\widetilde{\mathcal{P}}$,

$$\begin{aligned}\{\psi(A), \widetilde{J}_\xi\}^\sim(p, \nu) &= \{\psi(A) \circ i_\nu, \widetilde{J}_\xi \circ i_\nu\}(p) - \{\psi(A) \circ i_p, \widetilde{J}_\xi \circ i_p\}^{\mathcal{O}}(\nu) \\ &= \{A \circ \Phi_{s(\nu)^{-1}}, J_\xi\}(p) - \{F_p, \xi\}^{\mathcal{O}}(\nu).\end{aligned}$$

By Lemma 5(b) and (c), each of these terms is in $I(\widetilde{J})$ locally, and therefore globally by the partition of unity argument. □

Let Π be the projection from $N(I(\widetilde{J}))$ to $\mathcal{A}_S$. By Proposition 6, the composition $\Pi \circ \psi$ is defined. The map ψ depends on the choice of the set of sections $\mathcal{S}$: although $\mathcal{S}$ is chosen so that the transition functions take values in G_μ, the individual sections in general will not be so restricted. However, the next two results show that the composite $\Pi \circ \psi$ is independent of the choice of $\mathcal{S}$.

Proposition 7. *If $\widetilde{E} \in N(I(\widetilde{J}))$ and $\widetilde{E} \circ i_\mu \equiv 0$, then $\widetilde{E} \in I(\widetilde{J})$.*

Corollary 8. *The map $\Pi \circ \psi$ is independent of the choice of the set of sections $\mathcal{S}$ satisfying the conditions in Proposition* 4.

Proof of Corollary 8. Construct ψ as usual and construct $\widehat{\psi}$ in the same way but from a different set $\widehat{\mathcal{S}}$ of sections (which also satisfy the conditions in Proposition 4). Let $A \in N(I(K))^H$, and let $\widetilde{E} = \psi(A) - \widehat{\psi}(A)$. By Proposition 6, $\widetilde{E} \in N(I(\widetilde{J}))$. By Proposition 4(b), $s(\mu) = e = \hat{s}(\mu)$. Then

$$\widetilde{E} \circ i_\mu(p) = A(s(\mu)^{-1} \cdot p) - A(\widehat{s}(\mu)^{-1} \cdot p) = A(p) - A(p) \equiv 0.$$

Then by Proposition 7, $\widetilde{E} \in I(\widetilde{J})$; that is, $\psi(A)$ and $\widehat{\psi}(A)$ are equal mod $I(\widetilde{J})$ and thus $\Pi \circ \psi = \Pi \circ \widehat{\psi}$. □

Proof of Proposition 7. For a neighborhood $\mathcal{U} \subseteq \mathcal{O}$, define a function g on $[1, 0] \times \mathcal{U}$ to be "t-piecewise smooth" (in (t, ν)) if $[0, 1]$ can be divided into a finite number of closed subintervals $[t_i, t_{i+1}]$ such that g is smooth on every $[t_i, t_{i+1}] \times \mathcal{U}$. For every point ν_0 in $\mathcal{O}$, there is a neighborhood $\mathcal{U}$ of ν_0 and a continuous, t-piecewise smooth function g with values in G such that $g(0, \nu) = e$ and $g(1, \nu) \cdot \nu = \mu$. (For example, pick $\mathcal{U}$ to be convex and pick g so that $g(t, \nu) \cdot \nu$ shrinks $\mathcal{U}$ to a single point ν_0 as t runs from 0 to $\frac{1}{2}$. Then for $t \in \left[\frac{1}{2}, 1\right]$, pick $h(t)$ to be a path from the identity e in G_μ to a point in the coset corresponding to ν_0, and set $g(t, \nu) = h(t)^{-1} g\left(\frac{1}{2}, \nu\right)$.) Letting $g(t, \nu)$ act on $\widetilde{\mathcal{P}}$ creates a (continuous and t-piecewise smooth) time-dependent flow with generator

$$\frac{\partial}{\partial t}[g(t, \nu) \cdot (p, \nu)] = C^i(t, \nu) \xi_{i\widetilde{\mathcal{P}}}(g(t, \nu) \cdot (p, \nu))$$

for some t-piecewise smooth functions C^i. Then

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{E}(g(t,\nu)\cdot(p,\nu)) &= \mathrm{d}\widetilde{E}\left(\frac{\partial}{\partial t}[g(t,\nu)\cdot(p,\nu)]\right)\\ &= C^i(t,\nu)\,\mathrm{d}\widetilde{E}(\xi_{i\widetilde{P}}(g(t,\nu)\cdot(p,\nu)))\\ &= C^i(t,\nu)\{\widetilde{E},\widetilde{J}_i\}(g(t,\nu)\cdot(p,\nu)).\end{aligned}$$

But $\widetilde{E}\in N(I(\widetilde{J}))$ and $\widetilde{J}$ is equivariant, so there are t-piecewise smooth functions D^i on $[1,0]\times\widetilde{\mathcal{U}}$ such that

$$\frac{\partial}{\partial t}\widetilde{E}(g(t,\nu)\cdot(p,\nu)) = D^i(t,p,\nu)\widetilde{J}_i(p,\nu).$$

Note that

$$\widetilde{E}(g(1,\nu)\cdot(p,\nu)) = \widetilde{E}(g(1,\nu)\cdot p,\mu),$$

which by assumption is identically zero. Therefore

$$\begin{aligned}\widetilde{E}(p,\nu) &= \widetilde{E}(g(1,\nu)\cdot(p,\nu)) - \int_0^1 \frac{\partial}{\partial t}\widetilde{E}(g(t,\nu)\cdot(p,\nu))\,\mathrm{d}t\\ &= 0 - \int_0^1 D^i(t,p,\nu)\widetilde{J}_i(p,\nu)\,\mathrm{d}t = \left[-\int_0^1 D^i(t,p,\nu)\,\mathrm{d}t\right]\widetilde{J}_i(p,\nu).\end{aligned}$$

This shows that $\widetilde{E}$ is in $I(\widetilde{J})$ locally, and therefore globally by the partition of unity argument. □

The reduced algebra $\mathcal{A}_K$ is a quotient of the domain of $\Pi\circ\psi$ by the intersection of that domain with $I(K)$. It remains to show that $\Pi\circ\psi$ descends to a map on $\mathcal{A}_K$ with the desired properties.

Theorem 9. *The map ψ induces a well-defined Poisson algebra isomorphism*

$$\Psi:\mathcal{A}_K\to\mathcal{A}_S,$$

which is independent of the choice of the set $\mathcal{S}$ of sections satisfying the conditions in Proposition 4.

Proof. It suffices to verify the following statements.

(i) If $A\in[I(K)\cap N(I(K))]^H$ then $\psi(A)\in I(\widetilde{J})$. As ψ is linear, combined with Proposition 3 this implies that $\Pi\circ\psi$ induces a well-defined map $\Psi:\mathcal{A}_K\to\mathcal{A}_S$. Independence of the choice of $\mathcal{S}$ follows from Corollary 8.

(ii) The composite $\Pi\circ\psi$ is a Poisson map, so Ψ is a Poisson map.

(iii) If $\psi(A)\in I(\widetilde{J})$, then $A\in I(K)$, so Ψ is injective.

(iv) Given $\widetilde{A}\in N(I(\widetilde{J}))$, there is a function $A\in N(I(K))^H$ such that $\psi(A)$ is in the equivalence class of $\widetilde{A}$, so Ψ is also surjective.

For (i), note that $A = A^i K_i$ for some functions $A^i \in C^\infty(\mathcal{P})$, so by (8) $\psi(A)$ is given locally by

$$\psi(A)(p,\nu) = (A^i \circ \Phi_{s(\nu)^{-1}})(K_i \circ \Phi_{s(\nu)^{-1}}).$$

By Lemma 5(a), the latter factor is in $I(\widetilde{J})$ locally, so by the partition of unity argument, $\psi(A)$ is in $I(\widetilde{J})$. The rest of (i) follows as stated.

For parts (ii) and (iv), it is useful to pick a particular basis $\xi_1, \ldots, \xi_n$ for $\mathfrak{g}$. Recall that the orbit $\mathcal{O}$ is symplectic. For a given ν_0 in $\mathcal{O}$ choose the ξ_i for $i = 1, \ldots, m = \dim(\mathcal{O})$ to be canonical at ν_0; i.e. so that

$$\{\xi_i, \xi_j\}^{\mathcal{O}}(\nu_0) = \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}.$$

In particular the set of ξ_i will be a coordinate system on a neighborhood of ν_0. For $i > m$, choose ξ_i to belong to $\mathfrak{g}_{\nu_0}$, the stabilizer algebra of ν_0.

For the first clause of (ii), it suffices to show that if $A_1, A_2 \in N(I(K))^H$, then

$$\{\psi(A_1), \psi(A_2)\}^{\sim} - \psi(\{A_1, A_2\}) \in I(\widetilde{J}). \tag{10}$$

Note that $\{A_1, A_2\} \in N(I(K))^H$, because the action of H preserves the Poisson bracket, so the second term of (10) is well defined. This second term is

$$\psi(\{A_1, A_2\})(p,\nu) = \{A_1, A_2\}(s(\nu)^{-1} \cdot p).$$

The first term in (10) is given locally by

$$\begin{aligned} &\{\psi(A_1) \circ i_\nu, \psi(A_2) \circ i_\nu\}(p) - \{\psi(A_1) \circ i_p, \psi(A_2) \circ i_p\}^{\mathcal{O}}(\nu) \\ &\quad = \{A_1 \circ \Phi_{s(\nu)^{-1}}, A_2 \circ \Phi_{s(\nu)^{-1}}\}(p) - \{F_1, F_2\}^{\mathcal{O}}(\nu), \end{aligned} \tag{10a}$$

where $F_i(\nu) = A_i(s(\nu)^{-1} \cdot p)$. The action Φ preserves the Poisson bracket on $\mathcal{P}$, so the first term in (10a) cancels the second term in (10), and the problem reduces to consideration of $\{F_1, F_2\}^{\mathcal{O}}$. Fix a point ν_0 and pick a basis ξ_i as above. Let X be the Hamiltonian vector field for F_2. On a neighborhood of ν_0,

$$X = \sum_{i=1}^{m} X(\xi_i)\frac{\partial}{\partial \xi_i} = \sum_{i=1}^{m} \{\xi_i, F_2\}^{\mathcal{O}} \frac{\partial}{\partial \xi_i}$$

and therefore

$$\{F_1, F_2\}^{\mathcal{O}} = \mathrm{d}F_1(X) = \sum_{i=1}^{m} \{\xi_i, F_2\}^{\mathcal{O}}\, \mathrm{d}F_1(\partial/\partial \xi_i).$$

By Lemma 5(c), $\{\xi_i, F_2\}^{\mathcal{O}}$ is in $I(\widetilde{J})$ locally near ν_0. As ν_0 was arbitrary, the partition of unity argument shows that $\{F_1, F_2\}^{\mathcal{O}}$ is in $I(\widetilde{J})$. The Poisson structure on $\mathcal{A}_K$ descends from that on $N(I(K))$, so the final conclusion of (ii) follows.

In (iii), from $\psi(A) \in I(\widetilde{J})$ it follows that

$$\psi(A)(p,\nu) = \widetilde{A}^i(p,\nu)(J_i(p) - \langle \nu, \xi_i \rangle).$$

By Proposition 4(b), $s(\mu) = e$, so

$$A(p) = A(s(\mu)^{-1} \cdot p) = \psi(A)(p, \mu)$$
$$= \widetilde{A}^i(p, \mu)(J_i(p) - \langle \mu, \xi_i \rangle) = \widetilde{A}^i(p, \mu) K_i(p);$$

that is, $A \in I(K)$.

For statement (iv), the obvious candidate for A is $\widetilde{A} \circ i_\mu$; however, this function may not lie in $N(I(K))$, so it must be modified as follows. Pick a basis ξ_i as described above, but with $\nu_0 = \mu$. Note that for each $i \leq m$, there is one and only one $j = f(i)$ such that

$$\langle \mu, [\xi_{f(i)}, \xi_i] \rangle = \{\xi_{f(i)}, \xi_i\}^{\mathcal{O}}(\mu) = \pm 1,$$

and for other values of j,

$$\langle \mu, [\xi_j, \xi_i] \rangle = \{\xi_j, \xi_i\}^{\mathcal{O}}(\mu) = 0. \tag{11}$$

Also (11) holds for $i > m$: the generating vector field of the coadjoint action is [8]

$$(\xi_i)_{\mathcal{O}}(\mu) = -ad^*_\xi \mu = \langle \mu, [\ , \xi_i] \rangle = 0 \tag{12}$$

because $\xi_i \in \mathfrak{g}_\mu$ for all $i > m$. Let

$$C = \widetilde{A} \circ i_\mu + \sum_{j=1}^{m} A^j K_j, \tag{13}$$

where

$$A^j(p) = -\langle \mu, [\xi_{f(j)}, \xi_j] \rangle \{\widetilde{A} \circ i_p, \xi_{f(j)}\}^{\mathcal{O}}(\mu).$$

The computation below shows that this function C belongs to $N(I(K))$, and that averaging C over H gives the desired function A.

Using the equivariance of J and the fact that J and K differ by a constant, one computes that

$$\{K_j, K_i\}(p) = \langle J(p), [\xi_j, \xi_i] \rangle = \langle K(p) + \mu, [\xi_j, \xi_i] \rangle.$$

Thus by reasoning as in the proof of Proposition 6, we have

$$\{C, K_i\}(p) = \{\widetilde{A} \circ i_\mu, K_i\}(p) + \sum_{j=1}^{m} [\{A^j, K_i\} K_j + A^j \{K_j, K_i\}](p)$$
$$= \{\widetilde{A}, \widetilde{J}_i\}^\sim(\mu, p) - \{\widetilde{A} \circ i_p, \xi_i\}^{\mathcal{O}}(\mu)$$
$$+ \sum_{j=1}^{m} [\{A^j, K_i\} K_j + A^j \langle K, [\xi_j, \xi_i] \rangle + A^j \langle \mu, [\xi_j, \xi_i] \rangle](p).$$

The third and fourth terms are manifestly in $N(I(K))$. The first term belongs to $I(\widetilde{J}) \circ i_\mu = I(K)$, because $\widetilde{A} \in N(I(\widetilde{J}))$. If $i > m$, then the second and fifth terms vanish by (12). If

$i \leq m$, then the fifth term vanishes except for the summand with $j = f(i)$. Note that if $j = f(i)$, then $i = f(j)$, and

$$\langle \mu, [\xi_{f(j)}, \xi_j] \rangle = \langle \mu, [\xi_i, \xi_j] \rangle = -\langle \mu, [\xi_{f(i)}, \xi_i] \rangle = \pm 1.$$

Thus the fifth term reduces to

$$-\langle \mu, [\xi_{f(j)}, \xi_j] \rangle \{\widetilde{A} \circ i_p, \xi_{f(j)}\}^{\mathcal{O}}(\mu) \langle \mu, [\xi_{f(i)}, \xi_i] \rangle = +\{\widetilde{A} \circ i_p, \xi_i\}^{\mathcal{O}}(\mu),$$

which cancels the second term and thus shows that $C \in N(I(K))$.

Let $A = \bar{C}$. It remains to show that $\psi(A)$ and $\widetilde{A}$ belong to the same equivalence class, i.e. that $\psi(A) - \widetilde{A} \in I(\widetilde{J})$. By Proposition 3, $A \in N(I(K))$ and $A - C \in I(K)$. Then

$$A - \widetilde{A} \circ i_\mu = (A - C) + (C - \widetilde{A} \circ i_\mu) \in I(K)$$

by (13), so

$$(\psi(A) - \widetilde{A}) \circ i_\mu = A - \widetilde{A} \circ i_\mu = C^i K_i.$$

Define

$$\widetilde{E}(p, \nu) = (\psi(A) - \widetilde{A})(p, \nu) - C^i(p) \widetilde{J}_i(p, \nu).$$

Then $\widetilde{E} \circ i_\mu \equiv 0$. Also $\widetilde{E}$ is in $N(I(\widetilde{J}))$, because $\psi(A) \in N(I(\widetilde{J}))$ by Proposition 6, $\widetilde{A} \in N(I(\widetilde{J}))$ by assumption, and $\widetilde{J}_i \in N(I(\widetilde{J}))$ by equivariance. Thus by Proposition 7, $\widetilde{E} \in I(\widetilde{J})$. But

$$(\psi(A) - \widetilde{A} - \widetilde{E})(p, \nu) = C^i(p) \widetilde{J}_i(p, \nu) \in I(\widetilde{J}),$$

so $\psi(A) - \widetilde{A} \in I(\widetilde{J})$. □

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